A Characteristics Approach to Swept Shock-Wave/Boundary-Layer Interactions

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A theoretical study was made of shock/boundary-layer interaction in three dimensions. Using the small-perturbation approach of the Lighthill triple-deck model, the nature of the propagation of spanwise disturbances in swept interactions was analyzed for nonseparating flows. It was found that, to a first approximation, disturbances propagate upstream along shock/boundary-layer interaction characteristics, the direction of which is determined by the properties of the boundary layer and the mainstream. For a semi-infinite swept interaction, the analysis predicts that the flow adopts a generally cylindrically symmetric form and that the end effects are manifested by termination of the upstream flow pattern along a shock/boundary-layer interaction characteristic. Predicted pressure distributions are found to be consistent with experimental results for glancing shock interactions.

Nomenclature

a	= speed of sound
L	= inner deck, thickness, Eq. (13)
M	= Mach number
M_∞	= external flow Mach number
M_l	= component of mainstream Mach number normal to the interaction line
M_x	= component of wall-layer Mach number
$M_x(y)$	= component of boundary-layer Mach number normal to the interaction line
p	= pressure
p'	$= (p-p_0)/\gamma p_0$
$R(y)$	= basic flow density
s	= distance along interaction characteristic
$U(y), W(y)$	= basic flow velocities
u,v,w	= velocities in wall and boundary layers
u',v',w'	= perturbation velocities in x,y,z directions
x,y,z	= coordinates, Figs. 2 and 6
$\alpha(y)$	= basic flow speed of sound
β	$= \sqrt{M_l^2 - 1}$
γ	= ratio of specific heats
δ	= thickness of wall and boundary layers
ξ	= angle between interaction and characteristic lines
η	= deflection angle of boundary-layer streamlines
κ	= logarithmic decrement of upstream influence [Eq. (21)]
λ	= sweep angle, Fig. 2
ρ	= density

Subscripts

0	= upstream values
2	= values at interaction line
x	= values at line x = const
δ	= values at edge of boundary layer

I. Introduction

ALTHOUGH it has long been realized that interactions between shock waves and boundary layers very often involve three-dimensional effects, the difficulty involved in gaining an understanding of the problems posed by the two-dimensional interaction has inhibited study of the more complicated three-dimensional case. Nevertheless, sufficient experimental work has been done[1-4] to indicate that three-dimensional interactions may involve special effects not present in two-dimensional interactions and for which special explanations need to be developed. The spanwise development of a swept interaction appears to be one such effect.

As a frame of reference, this paper may be regarded as dealing with swept interactions of the two forms indicated in Fig. 1. Figure 1a shows what is often referred to as a "glancing interaction," in which a sharp flat plate is set at an angle of incidence with respect to an oncoming supersonic flow to generate an oblique shock normal to (and swept back along) the adjacent surface. The interaction of this shock with the boundary layer on that surface involves a swept interaction. Figure 1b shows a wedge attached to the surface on which the interaction is taking place, with the line of the corner between the wedge and the surface swept back with respect to the oncoming flow direction. The shock wave, or the corner between the wedge and the surface, defines an "interaction line" about which the interaction develops. Only "weak" interactions with small pressure rises are considered, so that the flow is always well removed from the separation condition.

If it were possible to set up a situation in which the boundary conditions applied by the shock wave on the corner were uniform along the interaction line, extending to infinity in both directions, then a cylindrically symmetric flow would be obtained. In such a flow, all the flow quantities are constant along lines parallel to the interaction line. However, in practice, the interaction must begin somewhere and this causes a change in the boundary conditions. The task is to determine how the change propagates in a spanwise direction along the interaction.

In order to determine the nature of the solution to this problem, a simple case is considered first, in which the boundary layer is represented as a uniform subsonic wall layer. The ideas developed in that analysis are then extended to the case of a real boundary layer. For this part of the analysis, the "triple-deck" model of the interaction is used. This model was originally developed by Lighthill[5] and applied by him to the calculation of the upstream influence of a weak interaction in a two-dimensional boundary layer. It has since

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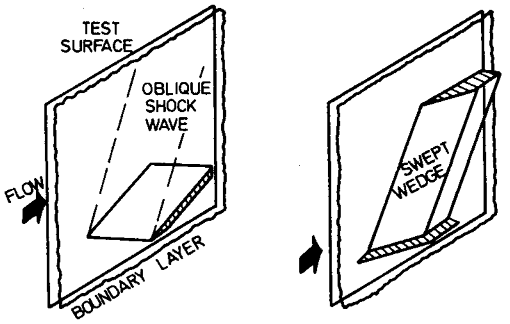


Fig. 1 Swept interaction configuration.

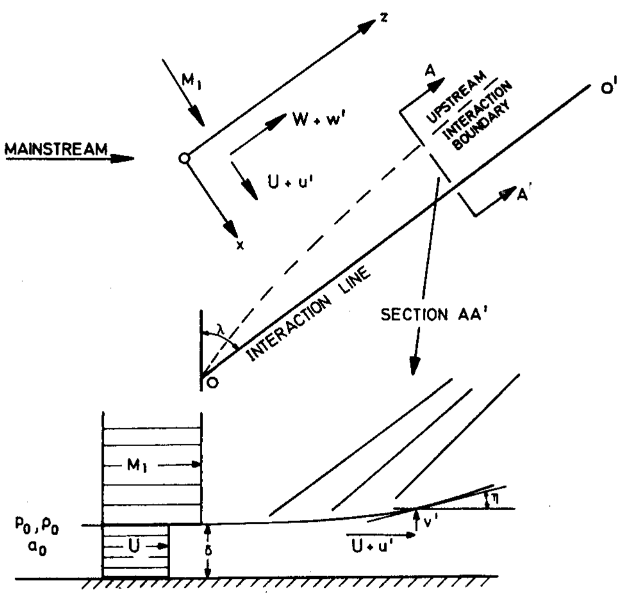


Fig. 2 Simplified interaction problem.

been used by other workers[6,7] in considering interactions strong enough for boundary-layer separation to take place and interactions involving turbulent boundary layers at high Reynolds numbers where the interaction lengths are small. In the present case, it is restricted to weak interactions, that is, where the boundary layer may be turbulent but is at a sufficiently low Reynolds' number that the interaction extends over at least a few boundary-layer thicknesses.

II. Analysis: Simplified Model

It is possible to model the essential physical features of a two-dimensional interaction by treating the boundary layer as a thin, uniform layer of subsonic flow between the wall and the supersonic mainstream. Although this model is not adequate for quantitative calculations of the upstream influence, it does display the exponentially decaying propagation of upstream influence characteristic of a weak shock/boundary-layer interaction.

Therefore, as a first step in understanding the physical features of a spanwise developing interaction, the simple model sketched in Fig. 2 is considered. The interaction is developed by applying a sudden pressure beginning at O on the interaction line OO′, which extends to infinity in the downstream spanwise direction. The interaction line is swept at an angle λ with respect to the oncoming mainstream flow direction. The thin wall layer (of thickness δ) is formed in the x,z plane and the component of Mach number (normal to the interaction line) of the flow in this layer (M_x) is subsonic, while the resultant direction of the flow in the wall layer is parallel to that of the mainstream. The pressure is constant across the wall layer. A cross section of the interaction, showing the upstream velocity profile and the thickening of the wall layer as the interaction progresses, is also displayed in Fig. 2.

Within the wall layer, velocity components in the x and z directions upstream of the interaction are U and W and the velocities within the interaction zone are taken to be $U+u'$ and $W+w'$, where u' and w' are small compared with U and W. Changes in pressure and temperature are also assumed to be small in relation to their upstream values. The fluid in the wall layer at the interface between the wall layer and the mainstream has a velocity v' normal to the wall and, in the x,y plane, the tangent to the interface makes an angle η with the wall. It follows that, to a first approximation, $\eta = v'/U$. Neglecting second-order terms, the equations of motion within the wall layer may be written as

$$U\frac{\partial u'}{\partial x} + W\frac{\partial u'}{\partial z} = -\frac{\partial p}{\partial x}\frac{1}{\rho_0} \tag{1a}$$

$$U\frac{\partial w'}{\partial x} + W\frac{\partial w'}{\partial z} = -\frac{\partial p}{\partial z}\frac{1}{\rho_0} \tag{1b}$$

for momentum in the x and z direction, and

$$\frac{-\eta U}{\delta} = \left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) + \frac{1}{\rho_0}\left(U\frac{\partial \rho}{\partial x} + W\frac{\partial \rho}{\partial z}\right) \tag{1c}$$

for the continuity relation, while

$$U\frac{\partial p}{\partial x} + W\frac{\partial p}{\partial z} = a_0^2\left(U\frac{\partial \rho}{\partial x} + W\frac{\partial \rho}{\partial z}\right) \tag{1d}$$

expresses the fact that the entropy remains constant along the streamlines. p_0, ρ_0, and a_0 are upstream values of pressure, density, and speed of sound, respectively.

Experimental results have shown that, in some cases, the dimensions of the interaction zone in the x direction tend to be much smaller than those in the z direction and that the interaction develops relatively slowly in the z direction. Considering only such cases for the moment, it is possible to make the convenient assumption that derivatives with respect to z are of order ϵ, in comparison with derivatives with respect to x where ϵ is small compared with unity. Thus, if $\partial u'/\partial x$ in Eqs. (1) is of $O(1)$, then it follows that $\partial u'/\partial z$ and $\partial p/\partial z$ are of $O(\epsilon)$ and $\partial w'/\partial z$ is of $O(\epsilon^2)$. Therefore, equations that are correct to order ϵ may be obtained by neglecting $\partial w'/\partial z$ and these equations may be combined to yield

$$\frac{\eta}{\delta} = (M_x^{-2} - 1)\frac{\partial p'}{\partial x} - \tan\lambda\frac{\partial p'}{\partial z} + U^{-1}\tan\lambda\frac{\partial u'}{\partial z} \tag{2}$$

where $p' = (p - p_0)/\gamma p_0$, $M_x = U/a_0$, and $\tan\lambda = W/U$.

An expression for $\partial u'/\partial z$ may be obtained† by referring to Eq. (1a), neglecting terms of order ϵ, and integrating the resulting first-order equation with respect to x. Noting that both u' and p' are zero far upstream of the interaction, this integration yields the relation

$$u' = -a_0^2 p'/U \tag{3}$$

and after differentiating with respect to z, this may be substituted into Eq. (2) to yield

$$\frac{\eta}{\delta} = (M_x^{-2} - 1)\frac{\partial p'}{\partial x} - (M_x^{-2} + 1)\tan\lambda\frac{\partial p'}{\partial z} \tag{4}$$

†Note that in Ref. 8, $\partial u'/\partial z$ was neglected, leading to the elimination of M_x^2 in the denominator of Eq. (5).

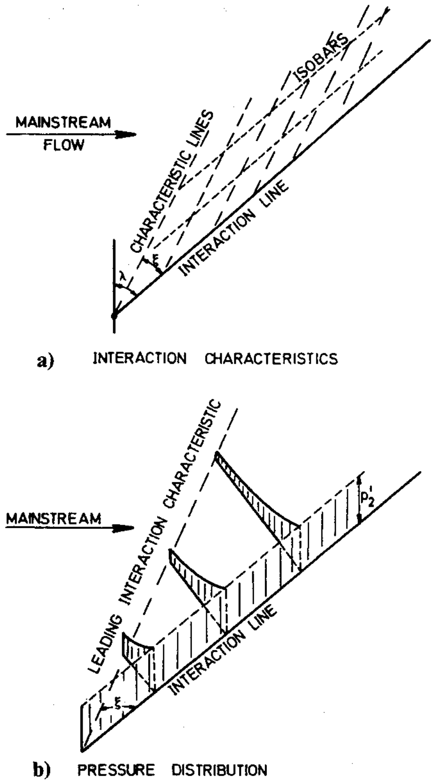


Fig. 3 Structure of interaction.

Now, the right-hand side of this equation states that the two partial derivatives of p', along x and z, may be combined into a single total derivative along lines whose slope is given by

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{M_x^{-2}-1}{M_x^{-2}+1}\frac{1}{\tan\lambda} \tag{5}$$

i.e., Eq. (4) is a compatability relation applying along characteristic lines that make an angle ξ with the interaction line, where

$$\tan\xi = -\frac{M_x^{-2}-1}{M_x^{-2}+1}\frac{1}{\tan\lambda} \tag{6}$$

Along these lines, p' is governed by the relation

$$\frac{\mathrm{d}p'}{\mathrm{d}s} = -\frac{\eta\sin\xi}{\delta(M_x^{-2}-1)} \tag{7}$$

where $\mathrm{d}s = -\mathrm{d}x\sin\xi + \mathrm{d}x\cos\xi$.

An approximate solution to Eq. (7) may be obtained by noting that, since the interaction develops relatively slowly in the z direction, the behavior of the mainstream approximates that of an infinite swept interaction and, neglecting terms of order ϵ, it is possible to write

$$\eta = p'\sqrt{M_1^2-1}\,/M_1^2 \tag{8}$$

which can be substituted into Eq. (7). The solution of the resulting equation satisfying the upstream boundary condition $p' \rightarrow 0$ as $s \rightarrow -\infty$ and the boundary condition $p' \rightarrow p_2$, at the interaction line, with $s=0$ at the interaction line, is

$$p' = p_2'\exp\{-s\sqrt{M_1^{-2}-1}\,\sin\xi/[M_1^2\delta(M_x^{-2}-1)]\} \tag{9}$$

Thus, p' decays exponentially in passing upstream along the interaction characteristics, and the value of p' on any characteristic is independent of the value on adjacent characteristics. In fact, p' is determined by the value of p_2' on the interaction line. The special (but usual) case where p_2' undergoes a step increase at the upstream end of the interaction line, and then remains constant, is shown in Fig. 3. Here the pressure is zero upstream of the leading interaction characteristic, which originates at the upstream end of the interaction line and, as indicated by Eq. (9), is constant along lines parallel to the interaction line. Thus, as shown in Fig. 3, the interaction is identical to the cylindrically symmetric one corresponding to the sweep angle λ terminated sharply along the leading interaction characteristic.

Physically, the form of the interaction field should not come as a surprise. If there is no wall layer, such that the flow is uniform all the way through to the wall, then a small pressure rise will propagate across the surface along a mainstream characteristic, i.e., at an angle equal to the Mach angle in the mainstream flow. If the Mach number in this flow is reduced, then the pressure rise propagates along a characteristic that is swept forward of its former position. If the Mach number is reduced only in a layer next to the wall, then the pressure rise persists in propagating along a characteristic that is forward of its former position but, as normally expected in a shock/boundary-layer interaction, the magnitude of the pressure rise is reduced as it passes upstream.

The simple model therefore demonstrates that the interaction field may be expected to contain two essential ingredients. One is that upstream propagation occurs along characteristics, the slope of which is dependent upon the boundary-layer properties, and the other is that the interaction decays exponentially along these characteristics. As in the case of two-dimensional flow, the rate of decay with upstream distance depends on the properties of both the wall layer and the mainstream.

III. The Mainstream Flow

With the form of the interaction field specified, it is possible to re-examine the assumption expressed in Eq. (8) that the mainstream flow behaves as if it were locally cylindrically symmetrical. This is done by taking the disribution of η as given by the solution and determining the nature of the associated mainstream pressure distribution at the outer edge of the wall layer.

Since the flow disturbance quantities are small, the mainstream behavior is described by linearized supersonic flow theory. The distribution of η, which is cylindrically symmetric downstream of the leading interaction characteristic and increases toward the interaction line, therefore may be replicated by superposition of an infinite series of flat supersonic wings, as shown in Fig. 4a. Each wing represents a small increment of η, i.e., $\Delta\eta$, and its planform differs from that of the previous wing in the series in that the leading edge, parallel to the interaction line, is moved downstream by an amount $\Delta\eta\,(\mathrm{d}x/\mathrm{d}\eta)$ corresponding to the associated $\Delta\eta$ in the interaction field.

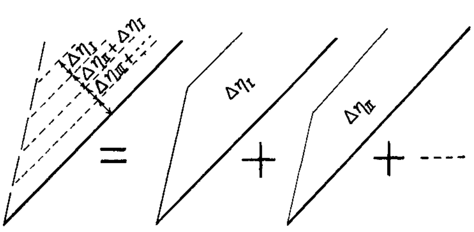
For each wing, the pressure distribution is as shown in Fig. 4b. In each of the panels ℓ and r, the pressure distribution is given by that for an infinitely swept wing, with leading-edge sweep angles of λ and $(\lambda-\xi)$, respectively. The associated pressures are given by

$$p_r - p_0 = \gamma p_0\Delta\eta M_\infty^2\cos^2\lambda(M_\infty^2\cos^2\lambda-1)^{-1/2} \tag{10}$$

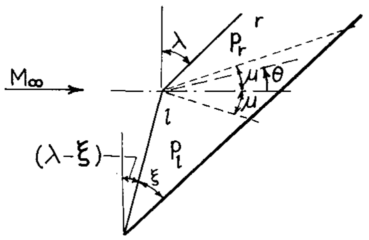
and

$$p_\ell - p_0 = \gamma p_0\Delta\eta M_\infty^2\cos(\lambda-\xi)\cos\lambda[M_\infty^2\cos^2(\gamma-\xi)-1]^{-1/2} \tag{11}$$

In between these two panels there is a region that, as shown on the figure, is delimited by the Mach lines originating at the junction of the two leading edges. A conical pressure



a) Superposition of flat wing solutions



b) Flat wing pressure pattern

Fig. 4 Mainstream pressure distribution.

distribution prevails in this region, which can be obtained by superposition of the pressure distributions presented in Ref. 9 with those for an infinite swept wing. This leads to the expression

$$p' = \Delta\eta\cos\lambda\frac{M_\infty^2}{\sqrt{M_\infty^2-1}}\left[\frac{m_\ell}{\sqrt{m_\ell^2-1}}+\frac{m_r}{\pi\sqrt{m_r^2-1}}\cos^{-1}\left(\frac{1-m_r\tau}{m_r-\tau}\right)-\frac{m_\ell}{\pi\sqrt{m_\ell^2-1}}\cos^{-1}\left(\frac{1-m_\ell\tau}{m_\ell-\tau}\right)\right] \quad (12)$$

where $m_r=\sqrt{M_\infty^2-1}\cot\lambda$ and $m_\ell=\sqrt{M_\infty^2-1}\cot(\lambda-\xi)$. τ is a conical variable, defined by $\tau=\sqrt{M_\infty^2-1}\tan\theta$, where, as shown in Fig. 4b, θ is the counterclockwise angle between the mainstream direction and a straight line through the apex of the conical region.

Provided ξ is small and (90 deg $-\lambda$) does not approach the Mach angle, Eqs. (10) and (11) show that the pressure difference between the two panels r and ℓ is of order ξ in comparison with the overall pressure rise on the wing. Since this is true for each of the wings in the series, it is also true when they are added together and leads to the conclusion that Eq. (8) describes the relation between pressure and deflection at the edge of the wall layer to within $O(\xi)$.

When (90 deg $-\lambda$) does approach the Mach angle, as for a glancing interaction, the two equations indicate that the increment in pressure from panel ℓ to panel r can be of the same order as the pressure itself. To obtain the pressure distribution, Eq. (12) can be used to plot the pressure distribution for each of the elemental wings noted above and these can then be added graphically to produce the resultant. Such distributions are shown in Fig. 5. These are presented for $M_\infty=2$, but will be typical of those obtained at other values of M_∞ in the supersonic range. Since η decays exponentially with the upstream distance, these pressure distributions prevail along any line $x=$ const, with $z=0$ taken at the leading interaction characteristic, and they may conveniently be referred to as $p'_{\eta x}$, the pressure obtained on that line according to Eq. (8). Noting that $\lambda=60$ deg along a Mach line, it can be seen that the pressure near the leading interaction characteristic drops to quite low values as (90 deg $-\lambda$) approaches the Mach angle. However, the gradient of pressure in the z direction remains small, being on the order of 1% of the pressure gradient in the x direction. This is an order of magnitude smaller than the smallest pressure gradient

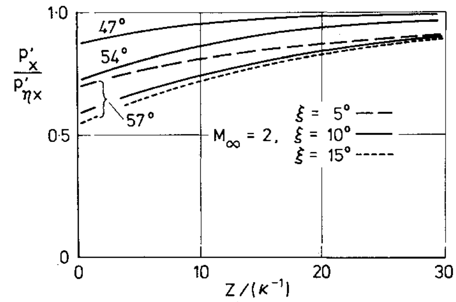


Fig. 5 Mainstream spanwise pressure variation with $\eta=\eta_x\exp(-s\kappa^{-1}\sin\xi)$.

retained in developing Eq. (4) and therefore this equation is unaffected by the spanwise gradient. Thus, the change in the relation between η and p' near the leading interaction characteristic does not change the angle ξ of the interaction characteristics, but it will somewhat change the upstream rate of decay of pressure in that region.

Anticipating the next section, it may be remarked that the arguments above will continue to apply to the mainstream when a more realistic boundary-layer model is used. Therefore, it remains a reasonable approximation to use the cylindrically symmetric form for the local pressure-deflection relation at the edge of the boundary layer, with the reservations noted in the previous paragraph.

IV. Analysis: Boundary-Layer Model

In the form outlined by Lighthill,[5] the "triple-deck" model of the interaction involves an external supersonic stream as an outer deck, with the boundary layer represented by a rotational inviscid flow as the middle deck, together with a viscous, incompressible flow as the inner deck. The middle deck occupies most of the boundary layer and, at least for interactions where the upstream interaction length is of the order of the boundary-layer thickness or greater, it incorporates the physical mechanisms that are primarily responsible for the interaction. The inner deck serves to damp the large velocity changes that otherwise would be induced close to the wall by the interaction and thereby determines the effective distance from the wall at which the flow takes on the inviscid character of the middle deck.

For two-dimensional flows, Lighthill derived a formula for this effective distance, i.e.,

$$L=0.78[\nu_w/\kappa U'(0)]^{1/3} \quad (13)$$

where L is the effective inner deck thickness, ν_w the kinematic viscosity at the wall, $U'(0)$ a normal gradient at the wall of the undisturbed boundary-layer velocity parallel to the wall, and κ the logarithmic decrement of upstream influence. The inverse logarithmic decrement is the distance, upstream of the interaction, over which the perturbation pressure falls by a factor of $e(=2.718)$, and therefore provides a good measure of the length scale of the upstream influence.

Equation (13) can readily be extended to apply to the cylindrically symmetric sweptback case,[4] with $U'(0)$ interpreted as the gradient of the component of velocity normal to the interaction line. Now, remembering that the experimental evidence indicates that the interaction develops relatively slowly in the spanwise direction, it follows that the inner layer flow will, to a first approximation, be identical to that in a locally cylindrically symmetric interaction and Eq. (13) [with appropriate $U'(0)$] will provide a first approximation to the thickness of the inner layer. Noting then that L varies only as the one-third power of κ, it is clear that L will vary only slowly as the upstream scale of the interaction increases, and therefore it is a reasonable approximation to assume that it remains constant at the value corresponding to a developed cylindrically symmetric interaction. Of course, this approximation will come to be substantially in error as one approaches the upstream end of the interaction, where the upstream influence is small. For a turbulent boundary layer, this is not likely to lead to serious error, provided that the scale of the upstream influence is at least as great as the order of magnitude of the boundary-layer thickness, since this scale is then affected by only a few percent by substantial variations in the inner layer thicknes. In the case of a laminar boundary layer, it could lead to significant error in the predicted development of the interaction close to the upstream corner.

With this approximation, the spanwise development of the interaction comes to be governed by the behavior of the flow in the middle deck. Following Lighthill, this is treated by allowing the small perturbations of the boundary-layer flow shown in Fig. 6a. In this flow, the velocity is unidirectional and parallel to the wall, varying only in the direction of y. $U(y)$, o, and $W(y)$ are the components of velocity in the x, y, and z directions, and $R(y)$, p_0, and $\alpha(y)$ are the density, pressure, and speed of sound. Then, assuming that $[u-U(y)]$, v, $[w-W(y)]$, $[\rho-R(y)]$, $[p-p_0]$, and $[a-\alpha(y)]$ are small, the continuity equation and the condition that the entropy remains constant along the streamlines may be combined to yield.

$$R(y)\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+R(y)\frac{\partial v}{\partial y}$$

$$=-\alpha(y)^{-2}\left[U(y)\frac{\partial p}{\partial x}+W(y)\frac{\partial p}{\partial z}\right] \tag{14a}$$

while the momentum equations in the x, y, and z directions can be written

$$R(y)\,U(y)\frac{\partial u}{\partial x}+R(y)\,W(y)\frac{\partial u}{\partial z}+R(y)\,vU'(y)=-\frac{\partial p}{\partial x} \tag{14b}$$

$$R(y)\,U(y)\frac{\partial v}{\partial x}+R(y)\,W(y)\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y} \tag{14c}$$

$$R(y)\,U(y)\frac{\partial w}{\partial x}+R(y)\,W(y)\frac{\partial w}{\partial z}+R(y)\,vW'(y)=-\frac{\partial p}{\partial z} \tag{14d}$$

where $U'(y)$ and $W'(y)$ signify differentiation with respect to y. As before, it is convenient to put $v=\eta U(y)$, implying that η again indicates the direction with respect to the surface of the component of velocity in the (x,y) plane. Remembering that $\alpha(y)^2=\gamma p_0/R(y)$ and $W(y)/U(y)=\tan\lambda$, Eqs. (14) then become

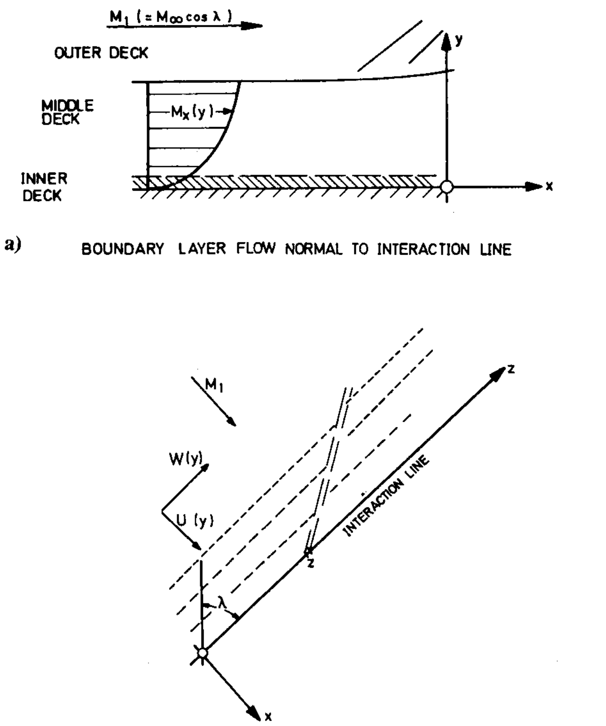
$$\frac{\partial\eta}{\partial y}=[M_x(y)^{-2}-1]\frac{\partial p'}{\partial x}-\tan\lambda\frac{\partial p'}{\partial z}$$

$$+\frac{\tan\lambda}{U(y)}\frac{\partial u}{\partial z}-\frac{1}{U(y)}\frac{\partial w}{\partial z} \tag{15a}$$

$$\frac{\partial u}{\partial x}+\tan\lambda\frac{\partial u}{\partial z}+\eta U'(y)=-U(y)M_x(y)^{-2}\frac{\partial p'}{\partial x} \tag{15b}$$

$$\frac{\partial\eta}{\partial x}+\tan\lambda\frac{\partial\eta}{\partial z}=-M_x(y)^{-2}\frac{\partial p'}{\partial y} \tag{15c}$$

$$\frac{\partial w}{\partial x}+\tan\lambda\frac{\partial w}{\partial z}+\eta\tan\lambda U'(y)=-U(y)M_x(y)^{-2}\frac{\partial p'}{\partial z} \tag{15d}$$



a) BOUNDARY LAYER FLOW NORMAL TO INTERACTION LINE

b) PROPAGATION OF SMALL DISTURBANCE

Fig. 6 Interaction with boundary layer.

where Eq. (15a) has been obtained by combining Eqs. (14a) and (14b).

It is now necessary to restrict our attention to the problem shown in Fig. 6b, in which a small change is made in the boundary condition at z on the interaction line—consisting, for example, of a small rise in the pressure—and one seeks to determine how this perturbation will propagate through the upstream influence region.

Now, Eq. (15c) embodies the effect of the lateral pressure gradient across the boundary layer from the wall to the mainstream. When the interaction length is an order of magnitude greater than the boundary-layer thickness (as, for example, in the case of a laminar boundary layer at a freestream Mach number of roughly 2 and with an adiabatic wall), one may justifiably assume that the pressure is constant across the boundary layer and integrate Eq. (15a) directly; Eq. (15c) then becomes superfluous. However, when the interaction lengths are "not so large," then the lateral pressure gradient can play a significant role, as it tends to cause a reduction in the pressure levels in the outer, supersonic layers of the boundary layer. Since these layers tend to contract laterally in response to a rising pressure in opposition to the expansion of the subsonic layers, they tend to reduce the upstream influence. If their contraction effect is reduced by reducing the pressure levels in the supersonic layers, then the upstream influence may be significantly increased. For this reason, it is desirable to include Eq. (15c) when dealing with cases, such as those involving turbulent boundary layers, where the interaction length is not very large.

However, most lateral expansion of the stream tubes will occur in the layers close to the wall, where the Mach number is low. Also, the most significant part of the lateral pressure change will occur in the outer layers of the boundary layer, where the velocities are highest. Therefore, as a rough approximation, it is assumed that the lateral pressure change takes place where the curvature of the boundary-layer streamlines is constant and equal to the value at the edge of the boundary layer. Further, since the disturbance to the cylindrically symmetric flowfield is assumed to be small, the curvature is taken to be equal to the undisturbed, cylindrically symmetric value, i.e., $(\partial\eta/\partial x)_\delta$. Equation (15c) therefore may be integrated from the edge of the boundary layer toward the surface to yield

$$p' = p'_\delta + \left(\frac{\partial\eta}{\partial x}\right)_\delta \int_y^\delta M_x(t)^2 \mathrm{d}t \tag{16}$$

This is now substituted into Eq. (15a), the resulting equation is integrated from the surface to the edge of the boundary layer, and the Prandtl-Meyer relation for the cylindrically symmetric mainstream flow, i.e.,

$$p'_\delta = M_I^2 \beta^{-1} \eta_\delta \tag{17}$$

is used to eliminate p'_δ. Then, assuming that the lateral pressure gradient is not large, it is accounted for approximately by using the first-order cylindrically symmetric solution, obtained by neglecting all but the first of the terms on the right-hand side of the equation, to yield

$$\left(\frac{\partial\eta}{\partial x}\right)_\delta \simeq \eta_\delta \left\{M_I^2\beta^{-1}\int_0^\delta [M_x(y)^{-2} - 1]\mathrm{d}y\right\}^{-1} \tag{18}$$

Substituting this into Eq. (15a), it finally becomes

$$\eta_\delta = \left\{A + A^{-1}\left[\int_0^\delta [M_x(y)^{-2} - 1]\left[\int_y^\delta M_x(t)^2\mathrm{d}t\right]\right.\right.$$
$$\left.\left.\times \mathrm{d}y\right]\right\}\left(\frac{\partial n_\delta}{\partial x}\right) - \tan\lambda\left\{M_I^2\beta^{-1}\delta + A^{-1}\int_0^\delta\left[\int_y^\delta M_x(t)^2\mathrm{d}t\right]\right.$$
$$\left.\times \mathrm{d}y\right\}\left(\frac{\partial n_\delta}{\partial z}\right) + \int_0^\delta [U(y)]^{-1}\left(\tan\lambda\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}\right)\mathrm{d}y \tag{19}$$

where A is the expression in brackets in Eq. (18).

Now, in treating the two-dimensional problem, Lighthill[5] shows that the dominant term in the expression describing the pressure distribution is of the form

$$p' = p(y)e^{\kappa x}$$

where κ is a constant. This relation, and that obtained with the simplified model in Sec. II, suggests that a solution be attempted of the form

$$p' = p(y)e^{\kappa x}G(\sigma) \tag{20a}$$

$$\eta = \eta(y)e^{\kappa x}G(\sigma) \tag{20b}$$

$$u = u(y)e^{\kappa x}G(\sigma) \tag{20c}$$

$$w = w(y)e^{\kappa x}G(\sigma) \tag{20d}$$

where $\sigma = z + x/\tan\xi$. Substituting these relations into Eqs. (19), (15b), and (15d), and employing the last two with Eqs. (16-18) to substitute for the last term in Eq. (19), a relation in $[G(\sigma)]^2$, $G(\sigma)$, $(\mathrm{d}G/\mathrm{d}\sigma)$, and $(\mathrm{d}G/\mathrm{d}\sigma)^2$ is obtained. If the disturbance introduced at z on the interaction line is such that flow quantities change only slowly along the interaction line, then $\mathrm{d}G/\mathrm{d}\sigma$ is small and terms in $(\mathrm{d}G/\mathrm{d}\sigma)^2$ may be neglected. Then it is found that Eq. (20) represents a solution to the problem provided that

$$\kappa = (A + B/A)^{-1} \tag{21‡}$$

‡Note that this reduces to Lighthill's expression for the logarithmic decrement of upstream influence[5] if the lower limit of the integral of $M_x(t)^2$ in Eq. (23b) is set equal to zero.

and

$$\tan\xi = (A + B/A)/(C + D/A) \tag{22}$$

where

$$A = M_I^2\beta^{-1}\int_0^\delta [M_x(y)^{-2} - 1]\mathrm{d}y \tag{23a}$$

$$B = \int_0^\delta [M_x(y)^{-2} - 1]\left[\int_0^\delta [M_x(t)^2\mathrm{d}t\right]\mathrm{d}y \tag{23b}$$

$$C = M_I^2\beta^{-}\tan\lambda\int_0^\delta [M_x(y)^{-2} + 1]\mathrm{d}y \tag{23c}$$

$$D = \tan\lambda\int_0^\delta [M_x(y)^{-2} + 1]\left[\int_y^\delta M_x(t)^2\mathrm{d}t\right]\mathrm{d}y \tag{23d}$$

Thus, as before, the solutions develop along the interaction characteristics. The small change in the boundary conditions at z in Fig. 6 will propagate along an interaction characteristic and the flow upstream and downstream of the characteristic will maintain a cylindrically symmetric form. The slope of the characteristic will be given by Eq. (22).

V. Evaluation for Turbulent Boundary Layer and Comparison with Experiment

For present purposes, it is sufficient to follow Lighthill and represent the Mach number profile in the turbulent boundary layer by the relation

$$M(y) = M_x(y)/\cos\lambda$$
$$= \{[M'(o)y]^{-2} + [M_\infty(y/\delta)^n]^{-2}\}^{-1/2} \tag{24}$$

Substituted into Eqs. (23), this yields the relations

$$A = M_I^2\beta^{-1}[\delta M_I^{-2}(1-2n)^{-1} - \delta] \tag{25a}$$

$$B = M_I^2\delta^2(1+2n)^{-1}$$
$$\times [M_I^{-2}(1-2n)^{-1} - 1 - 0.5M_I^{-2} + (2+2n)^{-1}] \tag{25b}$$

$$C = M_I^2\beta^{-1}[\delta M_I^{-2}(1-2n)^{-1} + \delta]\tan\lambda \tag{25c}$$

$$D = \delta^2[M_I^2(2+2n)^{-1} + 0.5(1-2n)^{-1}]\tan\lambda \tag{25d}$$

provided that $|\eta| < 0.5$, where terms involving $M'(o)$ have been omitted in Eqs. (25), since it is found that for a turbulent boundary layer they do not contribute significantly to the value of the expressions. Examination of Mach number profiles obtained at freestream Mach numbers of 4.5 and 2.7 in Ref. 10, together with measurements from Ref. 4 at a Mach number of 2.3, suggest that a value of $n = 0.25$ be used.

The analysis has been conducted for a small change in the boundary conditions on the interaction line but, since experimental data are available only for the propagation of the large change imposed by the beginning of the interaction, it is necessary to effect comparison with these data. Figure 7 shows comparisons of pressure distributions predicted by this theory with the experimental data for a glancing shock interaction (Fig. 1a) obtained by McCabe.[1]

For the comparisons, the average value of the pressure of the shock was taken from the measured pressure distribution and the theory was applied to calculate the expected pressure distribution upstream of the shock, as shown in Fig. 7. The boundary layer will not accept the infinite pressure gradient at the leading interaction characteristic implied by this pressure distribution, particularly near the upstream end of the interaction, and so the distribution there rapidly spreads up-

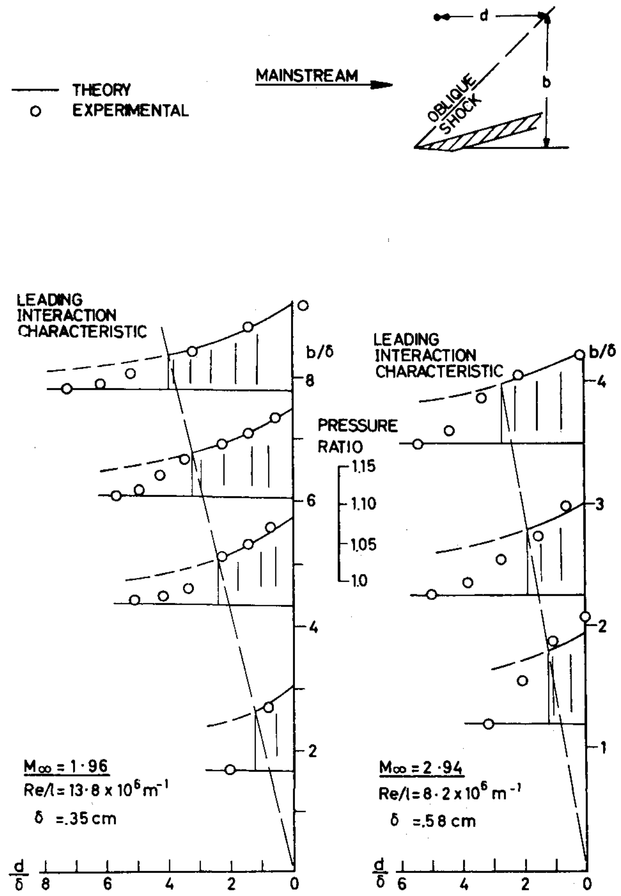


Fig. 7 Comparison with experiment.

stream of the interaction line over one or two boundary-layer thicknesses. This serves to eliminate the steep pressure gradients within a few spanwise boundary-layer thicknesses downstream of the beginning of the interaction, and the subsequent development of the interaction may be expected to take place along the interaction characteristics. These features are seen in the experimental results. The pressure field tends to depart from the cylindrically symmetric form along the predicted location of the leading interaction characteristic at both $M_\infty = 1.96$ and 2.94 and, at some distance from the upstream corner, the line along which a pressure rise is first experienced is also parallel to the leading interaction characteristic.

Some further comparisons can be gleaned from the exprimental results of other investigators. Dolling and Bogdonoff[2] used surface flow visualization at $M_\infty = 2.95$ to define the upstream boundary of the interaction between a glancing shock and the boundary layer on the floor of a supersonic tunnel. Taking their results at a shock deflection angle of 4 deg, which is sufficiently weak for the present theory to apply, it is found that the upstream boundary tends toward an angle of 8 deg with respect to the interaction line, whereas the characteristics angle predicted here is 6.2 deg. When the tunnel floor is replaced by a flat plate to allow the interaction to develop over three times as many boundary-layer thicknesses, this angle tends to a value of 6 deg. Peake[11] has also obtained pressure distributions with a similar configuration at $M_\infty = 2.0$. At an incidence of 4 deg, it is possible to use these pressure distributions to determine a characteristics angle of 9 deg in the upstream part of the interaction. This compares with a predicted value of 10.3 deg.

While these comparisons are encouraging, the data on which they are based are not sufficiently accurate to be conclusive. Further experiments are desirable, particularly involving more detailed pressure measurements than have been obtained in the past.

VI. Conclusion

The simplified analysis indicated that, for a swept shock/boundary-layer interaction with a small overall pressure rise, the flow generally adopts a cylindrically symmetric form. However, end effects, or changes in the boundary conditions along the interaction, are manifested by changes across shock/boundary-layer interaction characteristics. These propagate upstream at an angle determined by the sweep angle and the boundary-layer Mach number, but not the boundary-layer thickness.

Using the results of the simplified analysis, a solution was generated that was consistent with the more realistic description of the boundary-layer behavior afforded by the Lighthill triple-deck model. It was found that in this case, also, changes in the boundary conditions along the interaction were propagated upstream along shock/boundary-layer interaction characteristics, but that, although the angle of propagation was independent of the boundary-layer thickness, it depended on the sweep angle and the properties of both the boundary layer and the mainstream. This analysis was restricted to slowly varying spanwise disturbances but, provided that the mainstream continued to behave in a quasi-two-dimensional manner, the angle of propagation of the characteristics was not necessarily small. The analysis yielded results that were consistent with experiments involving a turbulent boundary layer.

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